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Invariant Measures and Convergence Properties for Cellular Automaton 184 and Related Processes

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Our results concern long time limit properties of a deterministic dynamics that is common for a wide class of processes that have been studied so far during at least last two decades. The most widely known process from this class is a cellular automaton that acquired number 184 in the classification of S. Wolfram. This CA 184 is being intensively used to model vehicular traffic. However, our results are mainly derived with help of another process that offers a helpful insight into the studied dynamics, it is a so-called Ballistic Annihilation Model (abbreviated by BA). BA is a model for chemical reaction $A + B \rightarrow$ inert. In BA, A and B-type particles move in opposite directions with velocities 1 and -1, respectively, and annihilate upon collisions. Certain results concerning BA and CA 184 are also formulated in terms of another process known as a Model of Surface Growth (SG, for short); the surface shape in this process behaves as the integrated profile of particle distribution in CA 184.

Our results are as follows. First, we characterize the invariant measures of the dynamics in interest. The bulk of our effort is devoted to the characterization of those of them that are not translation invariant; we call them phase separating invariant measures. In the case of BA, such measures are concentrated on the configurations consisting of two converging infinite blocks of (not necessarily adjacent) particles. In the case of CA 184, a phase separating measure describes the transition from free traffic phase to jammed phase. We also analyze domains of attraction of invariant measures and rates of convergence to them. This analysis then allows us to express the long time limit of particle current in CA 184 as a function of certain characteristics of its initial distribution, when it is translation invariant. This expression has been used in a companion paper (V. Belitsky, J. Krug, E. J. Neves and G. Schütz, A cellular automaton model for two-lane traffic, *J. Stat. phys.* **103**(5/6):945–971 (2001))

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to show the enhancement of cars' current caused by the possibility of lane changes in a model of traffic on a two-lane highway that was created by putting two CA 184's in parallel. Our other results concern hydrodynamic limits of BA and CA 184. We prove that if the integrated profile of initial particle configuration of BA or CA 184 converges, as $n \to \infty$, to some stochastic process $W(x), x \in \mathbb{R}$, when being re-scaled by n^{-1} along x-axis and by c_n^{-1} along y-axis for some sequence c_n, then the integrated profile of particle configuration at time n under the same re-scaling, will converge, as time $\to \infty$, to the local moving minimum of the process $W(\cdot)$, that is, to the process $W^{\min}(\cdot)$ defined by $W^{\min}(x) := \min\{W(y) : x-1 \leqslant y \leqslant x+1\}$. This hydrodynamic limit is then interpreted in terms of the limiting shape of surface in SG.

KEY WORDS: Cellular automata; Wolfram's automaton 184; ballistic annihilation; annihilating deterministic motions; surface growth; invariant measures; phase separating measures; hydrodynamic limits; rate of convergence to equilibrium; traffic flow models; flux of particles (cars); current of particles (cars).

1. INTRODUCTION

We study long time limit properties of a particular deterministic dynamics. This dynamics gave rise to a class of processes, called cellular automata, that have been studied so far in mathematical and physical literature, either separately or together. Three processes from this class "represent" the dynamics in our paper. Their constructions will be given in Section 2, while here we present only the motivation for our choice of these representatives. The first one of them has appeared in mathematical literature as a model for ballistic annihilation process; we thus call it BA. Besides of an independent interest in BA and its applications, the reason for its choice is in that it offers a very convenient insight into the studied dynamics: a reader will see that BA plays a central role in majority of our proofs. Our second "representative" is known under the name Cellular Automaton 184; we call it shortly by CA 184. It is being widely used for modeling traffic flow, which is the main reason for our interest in it. In particular, its properties studied in Section 5.3 were recently employed by us to construct and analyze a model of cars' traffic on a two-lane highway (see ref. 2). The third "representative" is the process known as Surface Growth Model; we call it SG. Its behavior is equivalent to that of the integrated profile of particle configurations of CA 184, and thus, SG appears naturally in the study of the hydrodynamic limit of CA 184 (see Section 4).

Our results are as follows: we characterize the invariant measures of the dynamics in interest (Section 3), and analyze the domain of attraction of the invariant measures and the rate of convergence to them (Section 5), and we also analyze hydrodynamic limits of particle distributions in BA and in CA 184 and the related limiting behavior of SG (Section 4).

We must note that certain ideas from our proofs may not be original, since the properties and applications of the studied dynamics have been investigated during at least last two decades (appropriate references will be given in the course of presentation). For example, our description of the translation invariant measures that are invariant measures for BA and for CA 184 may be derived by the tools that have appeared in previous studies (of Blank or of Gray and Griffeath, for example, or even an earlier work of Krug and Spohn); the same may be said in respect to some aspects addressed by us in Section 5. What is essentially novel in our results are: the description of the non-translation invariant measures that are invariant for the dynamics, and the identification of hydrodynamic limits of BA and CA 184. Also we believe we are original in our approach to analysis of the limit particle current in CA 184; the ideas of this approach were employed in ref. 2 mentioned above.

2. DEFINITIONS

In this section, we recall the constructions of the processes studies and give a brief historical note in respect to each one of them. As we have stated above, these processes are constructed on basis of a common dynamics. This fact is made precise by Lemmas 1 and 2 and Remark 1. Figure 1 illustrates the constructions and the lemmas.

2.1. Cellular Automaton 184

Cellular Automaton 184 (abbreviated by CA 184) is a discrete time process with state space $\{0,1\}^{\mathbb{Z}}$. Let, us usual, $\eta(x)$ denote the value of $\eta \in \{0,1\}^{\mathbb{Z}}$ at the coordinate $x \in \mathbb{Z}$. The CA 184 evolution rule acquires then the following definition: if $\eta \in \{0,1\}^{\mathbb{Z}}$ is the state at an arbitrarily fixed time n then $\hat{\eta} \in \{0,1\}^{\mathbb{Z}}$ defined by

$$\hat{\eta}(x) := \begin{cases} 1, & \text{if } \eta(x) = \eta(x+1) = 1 \\ 1, & \text{if } \eta(x) = 1 - \eta(x-1) = 0 \\ 0 & \text{otherwise} \end{cases} \qquad \forall\, x \in \mathbb{Z} \tag{1}$$

is declared to be the state at time $n+1$. Let the dynamics of CA 184 be denoted by the operator $C : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ such that $C\eta = \hat{\eta}$. Thus, formally, refers to a sequence $\{\eta_n, n \in \mathbb{Z}\}$ such that $\eta_n \in \{0,1\}^{\mathbb{Z}}$ and $\eta_{n+1} = C\eta_n \forall n$.

CA 184 models deterministic motions of identical particles on $\mathbb{Z}$ that obey the following rules: there may be at most one particle per site, and at each integer time each particle inquires whether the site of $\mathbb{Z}$ to the right

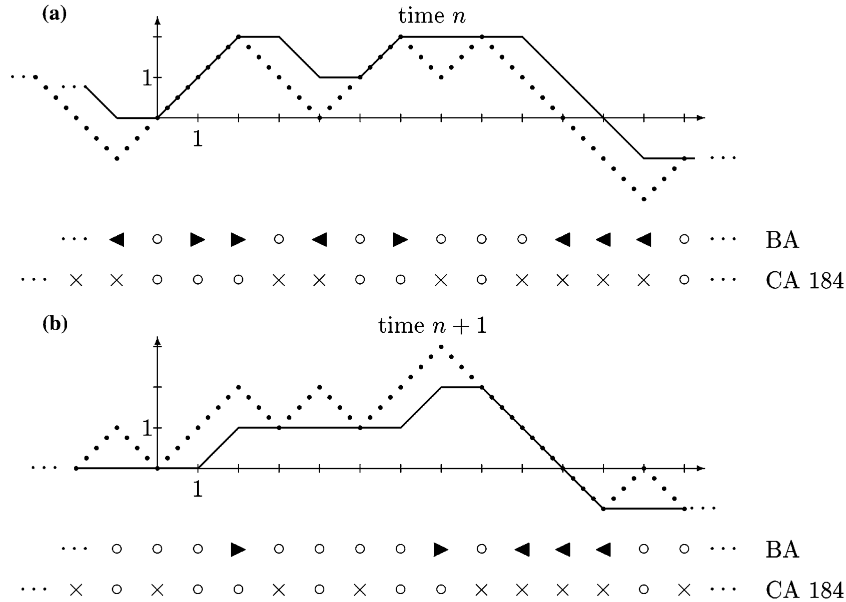

Fig. 1. Evolution of CA 184, BA and SG, construction of counting profiles for CA 184 and BA, and illustration of Lemmas 1 and 2.

In the bottom lines of figures (a) and (b), we present portion of configurations of CA 184 particles on $\mathbb{Z}$, where a site occupied by a particle is marked by $\times$ and a vacant site is marked by $\circ$. The configuration in (b) is obtained from that in (a) via the dynamics of CA 184. The upper lines in (a) and (b) present portion of configurations of BA particles on $\mathbb{Z}$, where a site containing negative (resp., positive) A-particle is presented by ◀ (resp., ▶), and an empty site is presented by $\circ$. The configuration in (b) is obtained from that in (a) via the dynamics of BA. The dotted curves in (a) and (b) present portions of functions from the space $\mathcal{R}$. The function from (b) is obtained from the function in (a) via the dynamics of SG.

In both (a) and (b), the configuration of BA particles relates to that of CA 184 ones via (4); the preservation of this relation in the passage from (a) to (b) illustrates the assertion of Lemma 1. On both (a) and (b), the dotted function relates to the configuration of CA 184 particles via (5), the preservation of this relation in the passage from (a) to (b) illustrates Lemma 2.

Solid curves in (a) and (b) are integrated profiles for the respective configurations of BA particles in the sense of (45). Dotted curves in (a) and (b) are integrated profiles for the respective configuration of CA 184 particles in the sense of (46). The figure illustrates the fact that the integrated profiles do not diverge one from another by more than 1, if both pass through $(0,0)$ and if they correspond to particle configurations that are related via (4); this fact is a cornerstone for the proof of Theorem 4.

of its current position is empty of another particle, and if it is so then it instantaneously jumps to this site. These rules follow immediately from (1), if one interprets $\eta(x)=1/0$ as "presence/absence of a particle at the site $x\in\mathbb{Z}$ in η".

The number "184" in the name of this process is due to the classification of Wolfram[21] (see also[22]) of a class of cellular automata. CA 184 has been employed for different needs. For example, in ref. 6 it is shown to be able to classify densities in binary strings. However, the most popular use of CA 184 and its divers modifications is for modeling vehicular flows; see in respect[2,12,19,20] (and the work of M. Blank submitted for publication in this journal in 2002) and references in the first two for the most recent achievements in this direction. CA 184 has two "stochastic" counterparts. One is the totally asymmetric simple exclusion process (TASEP). The second one has the same evolution rule as CA 184 but with "a noise" that is introduced by setting that each particle that can jump will do so with probability p independently of anything else; we thus call this process " with noise" (CA&N). The invariant measures for TASEP and CA&N have been characterized in refs. 18 and 23, respectively (curiously, only the case $p \leqslant 1/2$ was studied in ref. 23, but we believe that a similar technique allows to extend the results to any p) but the approach employed there (which is a stochastic coupling) does not apply to CA 184 (because of the lack of stochasticity in its dynamics). As we shall see, the set of the invariant measures for CA 184 is quite different from those for TASEP and for CA&N.

2.2. The Annihilating Particle System

The annihilating particle system studied here can be also found in the literature under the names Ballistic Annihilation (abbreviated by BA) and Annihilating Deterministic Motion. We shall adopt here the name BA. Let $\zeta(x)$ denote the value of $\zeta \in \{-1,0,1\}^{\mathbb{Z}}$ at the coordinate $x \in \mathbb{Z}$. BA is a discrete time process with state space $\{-1,0,1\}^{\mathbb{Z}}$ and the following dynamics: if $\zeta \in \{-1,0,1\}^{\mathbb{Z}}$ is the state at an arbitrarily fixed time n then $\hat{\zeta} \in \{-1,0,1\}^{\mathbb{Z}}$ defined by

$$\hat{\zeta}(x) = \begin{cases} 1, & \text{if } \zeta(x-1)=1 \text{ and neither } \zeta(x)=-1 \text{ nor both} \\ & \quad \zeta(x)=0, \quad \zeta(x+1)=-1 \\ -1, & \text{if } \zeta(x+1)=-1 \text{ and neither } \zeta(x)=1 \text{ nor both} \\ & \quad \zeta(x)=0, \quad \zeta(x-1)=1 \\ 0, & \text{otherwise} \end{cases} \qquad \forall x \in \mathbb{Z} \quad (2)$$

is declared to be the state at time $n+1$. The dynamics of BA will be denoted by the operator $A: \{-1,0,1\}^{\mathbb{Z}} \to \{-1,0,1\}^{\mathbb{Z}}$ such that $A\zeta = \hat{\zeta}$. Thus, formally, BA refers to a sequence $\{\zeta_n, n \in \mathbb{Z}\}$ such that $\zeta_n \in \{-1,0,1\}^{\mathbb{Z}}$ and $\zeta_{n+1} = A\zeta_n \forall n$.

BA may be also interpreted in terms of particles. We shall call them A-particles in order to distinguish them from those that move in

CA 184. The value $0, 1, -1$ of $\zeta(x)$ is interpreted by saying that site $x \in \mathbb{Z}$ is respectively, free of an A-particle, contains an A-particle with velocity 1, and contains an A-particle with velocity -1. In the terms of particles, the dynamics of BA acquires the following interpretation: each A-particle moves along $\mathbb{R}$ (exactly to state, along the line which contains the lattice $\mathbb{Z}$ on which the particles sit at integer times) with its velocity (going in the direction to $-\infty$ $(+\infty)$, if the velocity is negative (positive, resp.)) and annihilates when meets another A-particle; upon annihilation both A-particles disappear from the system forever.

BA is a natural model for the chemical reaction $A + B \rightarrow$ inert. For an extensive list of references in this respect, we refer a reader to.[7] A modification of BA in which the set of possible particle velocities is larger than $\{-1, 1\}$ has been considered in the works.[4,15] We also mention here the work[8] which studies a coalescing particle system employing a very simple relation between it and BA. This coalescing particle system is a process in which particles move as in BA but coalesce instead of annihilate, upon collision; coalesced particles choose then a new velocity from $\{-1, 1\}$ with equal probabilities.

2.3. The Surface Growth Model

The surface growth model (abbreviated by SG) is a discrete time process with state space $\mathcal{R}$, the space of piecewise linear functions (from $\mathbb{R}$ to $\mathbb{R}$) that have the slope of either 45° or 315° between any two consequent integer abscissas, and that attain integer value at each integer abscissa (see Fig. 1). Its dynamics is as follows. If $f(\cdot) \in \mathcal{R}$ is the state at an arbitrarily fixed time n then $\hat{f}(\cdot) \in \mathcal{R}$ defined by

$$\hat{f}(i) := \begin{cases} f(i) + 2, & \text{if } f(i) = f(i+1) - 1 = f(i-1) - 1 \\ f(i), & \text{otherwise} \end{cases} \qquad \forall i \in \mathbb{Z} \qquad (3)$$

is declared to be the SG state at time $n+1$ (certainly, to define $\hat{f} \in \mathcal{R}$, it is sufficient to determine its values at $\mathbb{Z}$). The dynamics of SG will be denoted by the operator $S : \mathcal{R} \rightarrow \mathcal{R}$ such that $Sf = \hat{f}$. Thus, formally, SG refers to a sequence $\{f_n(\cdot), n \in \mathbb{Z}\}$ such that $f_n \in \mathcal{R}$ and $f_{n+1}(\cdot) = Sf_n(\cdot) \forall n$.

Imagine $f(\cdot) \in \mathcal{R}$ as the surface of a two-dimensional solid above some reference horizontal line. Imagine then that diamond shaped particles of the side length $\sqrt{2}$ are thrown on this solid, and those of them that fall in local minima of $f(\cdot)$ stick to the solid while others disappear. The surface of the new solid will be then what we have defined as $\hat{f}(\cdot)$. This justifies the name given to the process. We observe that this process is also known under the name polynuclear growth model (PNG) and one of its modifications was studied almost 30 years ago in ref. 5. The ref. 17 contains a list of works in which this process has been addressed.

2.4. Equivalence of the Dynamics

Lemmas 1 and 2 and Remark 1 in this section give the precise meaning to equivalence of the dynamics of BA, CA 184 and SG. We note that this equivalence has been known since at least the work.[16] The assertions are illustrated in Fig. 1. Their proofs are straightforward and thus, omitted. We recall that the operators A, C and S, used below, denote the dynamics of BA, CA 184 and SG, respectively.

Lemma 1. **(Relation of CA 184 to BA).** *Define* $T_{184,\mathrm{BA}} : \{0,1\}^{\mathbb{Z}} \to \{-1,0,1\}^{\mathbb{Z}}$ *by*

$$(T_{184,\mathrm{BA}}\eta)(i) = 1 - \eta(i) - \eta(i-1), \quad i \in \mathbb{Z} \tag{4}$$

Let $\eta, \hat{\eta} \in \{0,1\}^{\mathbb{Z}}$ *and* $\zeta, \hat{\zeta} \in \{-1,0,1\}^{\mathbb{Z}}$ *be such that* $T_{184,\mathrm{BA}}(\eta) = \zeta$ *and* $T_{184,\mathrm{BA}}(\hat{\eta}) = \hat{\zeta}$. *Then,* $\hat{\eta} = C\eta$ *if and only if* $\hat{\zeta} = A\zeta$.

Lemma 2. **(Relation of SG to CA 184).** *Define* $T_{\mathrm{SG},184} : \mathcal{R} \to \{0,1\}^{\mathbb{Z}}$ *by*

$$\left(T_{\mathrm{SG},184} f\right)(i) = \begin{cases} 0, & \text{if } f(i) - f(i-1) = 1 \\ 1, & \text{if } f(i) - f(i-1) = -1 \end{cases} \qquad \forall i \in \mathbb{Z} \tag{5}$$

Let $f, \hat{f} \in \mathcal{R}$ *and* $\eta, \hat{\eta} \in \{0,1\}^{\mathbb{Z}}$ *be such that* $T_{\mathrm{SG},184} f = \eta$ *and* $T_{\mathrm{SG},184} \hat{f} = \hat{\eta}$. *Then,* $\hat{\eta} = C\eta$ *if and only if the functions* $\hat{f}$ *and* Sf *have the same shape, that is, if and only if* $(Sf)(x) - (Sf)(0) = \hat{f}(x) - \hat{f}(0)$, *for all* $x \in \mathbb{R}$.

Remark 1. Say that $f, g \in \mathcal{R}$ have the same *shape*, if $f(x) - f(0) = g(x) - g(0)$ for all $x \in \mathbb{R}$, and separate $\mathcal{R}$ in classes of equivalence by postulating that two functions with the same shape belong to the same class. It is easy to check that $T_{\mathrm{SG},184}$ induces a bijection between the state space of CA 184 and the classes of equivalence. This bijection and Lemma 2 allow one to reformulate any result about the CA 184 dynamics in terms of the SG dynamics on the classes of equivalence. A harder problem to which we have no generic treatment is to deduce how a single function is modified in SG from analysis of the relating CA 184 or BA. Let us give an example. Let f_0 be a random function such that $f_0(0) = 0$ and $\{f_0(i) - f_0(i-1),\, i \in \mathbb{Z}\}$ are i.i.d. Bernoulli 1/2 random variables. According to Lemmas 1 and 2, the distribution of A-particles relating to f_0 is $\zeta_0 = T_{184,\mathrm{BA}} \circ T_{\mathrm{SG},184} f_0$. Consider BA starting from this distribution. Using the methods developed in 1, it is possible to deduce approximately its distribution at an arbitrary time n. This however, is not sufficient to deduce the distribution of f_n. The reason is that on one hand, the distribution of BA at time n reveals solely the distribution of the shape of f_n (via $T^{-1}_{\mathrm{SG},184} \circ T^{-1}_{184,\mathrm{BA}}$) but not of the *height* of f_n (by which we mean $f_n(0)$), and, on the other hand, we do not have a complete control on the relation between the shape change sub-process and the height growth sub-process of SG. Thus, when we formulate a result about SG $\{f_n, n \in \mathbb{Z}\}$ deducing it from a result about BA, we resort to an implicit random variable that "shifts a function to where it is needed" without specifying the "needed" quantity. An example of how we do this, is provided by Theorem 5. In a similar way, one could obtain counterparts of Theorems 2, 6 and 7 for the SG process.

3. INVARIANT MEASURES

3.1. The set of the invariant measures for BA

The set of the invariant measures for BA is characterized in Theorem 1 of this section, where by "characterize" we mean that the theorem provides convenient tools that allow one to construct and to investigate measures from this set; in Remarks 2–4 we shall specify why we think these tools are convenient. As one may see, the bulk of the proof is devoted to treat the *phase separating* measures, i.e., the measures that are supported by the configuration set Θ_s to be defined below in (6). The set of phase separating measures that are invariant for BA is characterized via establishing that it is isomorphic to a particular set of almost stationary real valued processes. We precede the theorem formulation by constructions of the isomorphism (F_0^*) and the process distribution set ($\mathbf{E}_\sigma$). Figure 2 illustrates these constructions.

Elements from $\{-1, 0, 1\}^{\mathbb{Z}}$ will be called *configurations*, and their values at sites of $\mathbb{Z}$ will be interpreted in terms of particles as indicated in Section 2. An A-particle (from BA) whose velocity is $+1$ (resp., -1) will be called *positive* (resp., *negative*). A-particles will be called simply *particles*, when this does not create a confusion.

Two particles will be called *consecutive*, if there are no any other particle between them. A pair of consecutive A-particles is said to be *converging* (resp., *diverging*), if the leftmost (resp., rightmost) particle of the pair is positive and the rightmost (resp., leftmost) one is negative. Note that the terms converging and diverging apply solely to a pair of consecutive particles. We introduce

$$
\begin{aligned}
\Theta_s := \{\zeta \in \{-1,0,1\}^{\mathbb{Z}} : \zeta \ &\text{is a } \textit{phase separating} \text{ configuration, i.e.} \\
&\text{(i) there is only one pair of (consecutive) converging particles;} \\
&\text{(ii) both the number of positive particles in } \zeta \\
&\quad \text{and the number of negative ones are infinite}\}
\end{aligned} \tag{6}
$$

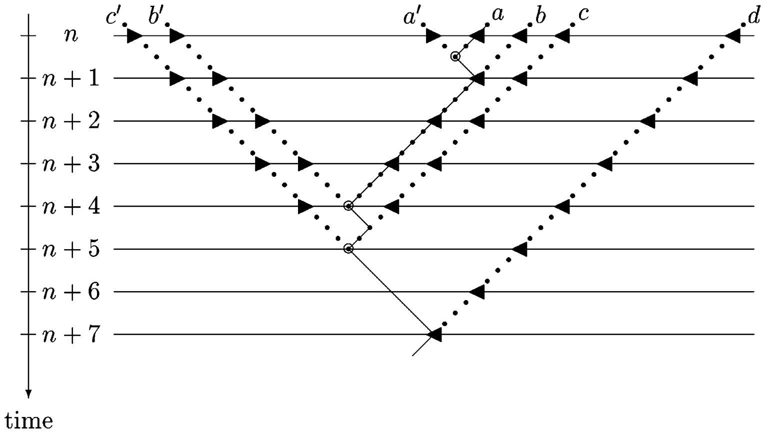

Fig. 2. The portion of $e \equiv \{(e_i, i), i \in \frac{1}{2}\mathbb{Z}\}$ from E' contained in time interval $[n, n+7]$ is represented in the figure by zig-zag line. All kinks of this portion of e are circled. Two rays are stemmed from each kink, one goes north-east and the other one north-west; the rays are indicated by dot lines (we note that the lowest dot-line comes from some kink below the level $n+7$). At the intersections of these rays with horizontal lines, we put A-particles: a positive particle (▶), if the ray involved goes north-east, and a negative particle (◀), if it goes north-west. The configuration of particles on the line at time-level n is what we have denoted by $F_n(e)$ in the proof of Theorem 1.

Let us interpret ▶s and ◀s in the figure as a portion of the "history" of BA, and let us exhibit how the second class particle would move in this portion of BA. According to the rules *(i)–(iii)* from the proof of Theorem 1, the second class particle moves together with particle a during time interval $[n, n+\frac{1}{2}]$, then, at time $n+\frac{1}{2}$, when a is annihilated by a', the second class particle moves together with the ghost of a' (this ghost is an inert particle that moves after time $n+\frac{1}{2}$ exactly as a' would moved, if it had not been annihilated at this time) until it meets particle b; from this moment till the annihilation time of b, the second class particle moves together with it, and starting from time $n+4$, it moves together with the ghost of b'; and so forth...

The figure shows clearly that if we construct BA process from $e \in E'$ then the second class particle trajectory in this BA will coincide with e. This is the cornerstone of the bijective relation between trajectories of the second class particle and "histories" of BA. This bijection in turn, is the cornerstone of assertion (b) of Theorem 1.

Let us finally illustrate that the mentioned above bijective relation would be broken if we index elements from E' by the time set $\mathbb{Z}$ rather than by $\frac{1}{2}\mathbb{Z}$. Take the portion of the zig-zag line between the levels $n+4$ and $n+5$ and reflect around the vertical line that links the kinks at $n+4$ and $n+5$. Observe that although the new zig-zag line, e', coincides with the old one, e, at integer "times" $\dots, n, n+1, \dots, n+7, \dots$, nevertheless the kinks of e at times $n+4$ and $n+5$ are not present in e'. Consequently, e and e' would generate different BA histories.

Note that if $\zeta \in \Theta_s$ then it contains only positive (resp., negative) particles to the left (resp., right) of the middle point between the (unique!) pair of its converging particles. This structure motivated the name we have given to Θ_s.

For brevity sake, we introduce symbol

$$\tfrac{1}{2}\mathbb{Z} := \mathbb{Z} \cup \left(\mathbb{Z} + \tfrac{1}{2}\right) \tag{7}$$

and introduce then a particular set of numeric sequences indexed by $\frac{1}{2}\mathbb{Z}$:

$$E := \left\{ e = \left\{e_i,\ i \in \tfrac{1}{2}\mathbb{Z}\right\} : e_i \in \mathbb{Z}\ \forall i \in \mathbb{Z},\ \text{ and } e_i - e_{i+\frac{1}{2}} \in \{-\tfrac{1}{2}, \tfrac{1}{2}\}\ \forall i \in \tfrac{1}{2}\mathbb{Z} \right\} \tag{8}$$

By *geometric representation* of $e \in E$ we shall call the set of points $\{(e_i, i), i \in \frac{1}{2}\mathbb{Z}\}$ in Euclidean plane whose horizontal axis corresponds to the first coordinate and its unit vector points rightwards, and whose vertical axis corresponds to the second coordinate and its unit vector points downwards; the first coordinate will be interpreted as position and the second as time (in the interacting particle system field, there is a tradition to orient the time axis downwards). We say that $e \in E$ has a *kink* at time $t \in \frac{1}{2}\mathbb{Z}$, if $e_{t-\frac{1}{2}} > e_t$ and $e_{t+\frac{1}{2}} > e_t$, and we define then

$$E' := \{ e \in E : e \text{ has infinitely many kinks after any time } t \} \tag{9}$$

Note that if t and t' are two consecutive kink times of an $e \in E$ and if (e_t, t) and $(e_{t'}, t')$ are given, then there is a unique choice for the values of e_u at $t \leqslant u \leqslant t'$; this fact follows easily from the geometric representation of e and implies the following property that will be used in our arguments:

$$\begin{gathered}\text{any } e \in E' \text{ is uniquely determined by the time}\\ \text{and space coordinates of all its kinks}\end{gathered} \tag{10}$$

We construct a mapping $F_0 : E' \to \{-1, 0, 1\}^{\mathbb{Z}}$ in the following way. Given $e \in E'$, we draw its geometric representation in Euclidean plane, and then, from each kink point (e_t, t) whose time coordinate is strictly (!) greater than 0, we emerge two rays on this plane, one going north-west and the other one going north-east. We then postulate that the particle configuration $F_0(e)$, that results from the application of F_0 to e, contains negative (resp., positive) A-particle at $k \in \mathbb{Z}$, if and only if $(0, k)$ is an intersection point of the horizontal axis with one of the rays that go north-east (resp., north-west) (see Fig. 2 for illustration).

Let us now introduce "the time shift of E by 1" as the mapping

$$\sigma : E \to E \text{ defined by } (\sigma(e))_n = e_{n-1} \quad \forall n \in \tfrac{1}{2}\mathbb{Z} \tag{11}$$

Let us then denote by $\mathbf{E}_\sigma$ the set of the measures on E' that are invariant with respect to the time shift by 1 (note, by 1 and not by $\frac{1}{2}$), that is, $\nu \in \mathbf{E}_\sigma$ iff $\nu\big(\sigma^{-1}U\big) = \nu\big(U\big)$ for any set $U \subset E'$. For a measure $\nu \in \mathbf{E}_\sigma$, let $F_0^*(\nu)$ denote the measure on Θ_s such that

$$F_0^*(\nu)\,(U) = \nu\left(F_0^{-1}(U)\right) \text{ for every cylinder } U \subseteq \Theta_s \subset \{-1,0,1\}^{\mathbb{Z}} \quad (12)$$

To see that $F_0^*(\nu)$ is well defined by (12), fix arbitrarily a natural number k and $2k$ integer numbers $\ell_k < \ell_{k-1} < \cdots < \ell_1 < r_1 < r_2 < \cdots < r_k$, and consider the set U composed of those ζ's from Θ_s that have positive particles at the fixed ℓ's and negative particles at the fixed r's, and no other particles in between ℓ_k and r_k. It is then follows from our construction of F_0 that $F_0^{-1}(U)$ consists of those e's that contain k kinks in the time interval $[0,(r_k-\ell_k)/2]$ whose positions are rigidly determined by the requirement that the i-th kink should generate the particles at the sites ℓ_i and r_i. Then, $\nu(F_0^{-1}(U))$ is well defined. The above considerations imply easily that (12) is well posed for any cylinder U, and then that $F_0^*(\nu)$ is a measure on Θ_s.

Theorem 1. (**Characterization of the invariant measures for BA**). Let ϕ denote the empty configuration, i.e., the configuration that has no particles, and let δ_ϕ denote the measure concentrated on ϕ. Let $\mathbf{I}^{\text{BA}}$ denote the set of the measures that are invariant for BA.

(a) A measure μ on $\{-1,0,1\}^{\mathbb{Z}}$ is invariant for BA if and only if

$$\mu = \alpha_1\mu_1 + \alpha_2\mu_2 + \alpha_3\mu_3 + \alpha_4\delta_\phi \quad (13)$$

for some non-negative real numbers $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ such that $\alpha_1+\alpha_2+\alpha_3+\alpha_4=1$, and some measures μ_1,μ_2,μ_3 from the respective sets $\mathbf{P}^{\text{BA}}_{+,\tau}$, $\mathbf{P}^{\text{BA}}_{-,\tau}$ and $\mathbf{P}^{\text{BA}}_s \cap \mathbf{I}^{\text{BA}}$ described below. The expansion (13) is unique.

$\mathbf{P}^{\text{BA}}_{+,\tau}$ and $\mathbf{P}^{\text{BA}}_{-,\tau}$ consist of the measures that are translation invariant, i.e., invariant with respect to the "the space shift of $\{-1,0,1\}^{\mathbb{Z}}$ by 1" operator

$$\tau:\{-1,0,1\}^{\mathbb{Z}} \to \{-1,0,1\}^{\mathbb{Z}} \text{ defined by } (\tau(\zeta))(x) = \zeta(x-1) \quad \forall x \in \mathbb{Z} \quad (14)$$

and are concentrated on the respective configuration sets

$$\begin{aligned}\Theta_+ &:= \{\zeta \in \{-1,0,1\}^{\mathbb{Z}} : \zeta \neq \phi, \text{ and } \zeta \text{ has no negative particles}\}\\ \Theta_- &:= \{\zeta \in \{-1,0,1\}^{\mathbb{Z}} : \zeta \neq \phi, \text{ and } \zeta \text{ has no positive particles}\}\end{aligned} \quad (15)$$

The set $\mathbf{P}^{\text{BA}}_s$ consists of the measures concentrated on the configuration set Θ_s defined in (6). It does not contain translation invariant measures.

(b) The transformation F_0^* defined in (12) is a one-to-one mapping between $\mathbf{E}_\sigma$ and $\mathbf{P}_s^{\mathrm{BA}} \cap \mathbf{I}^{\mathrm{BA}}$, the subset of $\mathbf{P}_s^{\mathrm{BA}}$ consisting of the measures invariant for BA.

Remark 2. Note that the construction of the measure set $\mathbf{P}_{+,\tau}^{\mathrm{BA}} \cup \delta_\phi$ implies directly that it is equivalent to the set of stationary processes with the time set $\mathbb{Z}$ and the state space $\{0, 1\}$. Similarly, $\mathbf{P}_{-,\tau}^{\mathrm{BA}} \cup \delta_\phi$ is equivalent to the set of stationary processes with the time set $\mathbb{Z}$ and the state space $\{0, -1\}$. Note also that the measure set $\mathbf{E}_\sigma$ may be viewed as a set of "almost stationary" processes whose trajectories belong to the set E' that has been defined in (9); the proviso "almost" is needed since the processes are indexed by the time set $\frac{1}{2}\mathbb{Z}$, but are invariant with respect to the time shift by 1, and not by 1/2. Thus, we say that Theorem 1 *characterizes* the invariant measures for BA since it allows one to construct and to study these measures with help of the theory of stationary processes, which is a well developed field.

Remark 3. The reader will see that the fact that F_0^* is a bijection is proved with help of so-called second class particle. Methods that use second class particle showed to be extremely effective in studying various interacting particle systems (see review paper 9), and in particular, has been used together with cellular automaton in ref. 10. The ideas employed here are illustrated in Fig. 2.

Remark 4. It follows from the structure of phase separating configurations (defined in (6)) that if a measure μ is supported by Θ_s then it may be identified with collection of distribution functions $(\mathcal{F}, \{\mathcal{F}_{r,x}, x \in \frac{1}{2}\mathbb{Z}\}, \{\mathcal{F}_{\ell,x}, x \in \frac{1}{2}\mathbb{Z}\})$, where $\mathcal{F}$ is a distribution on $\frac{1}{2}\mathbb{Z}$ that determines the position of the middle point between the converging particles, and $\mathcal{F}_{r,x}$ and $\mathcal{F}_{\ell,x}$ determine the positions of negative and positive particles respectively, to the right and to the left of x, given the middle point position is x. We tried to characterize the measure set $\mathbf{P}_s^{\mathrm{BA}} \cap \mathbf{I}^{\mathrm{BA}}$ with the help this identification, but the results we obtained were less satisfactory than those obtained by the second class particle method, when we compared them from the point of view of their applicability for constructing and analyzing the measures from $\mathbf{P}_s^{\mathrm{BA}} \cap \mathbf{I}^{\mathrm{BA}}$.

Proof of Theorem 1(a). Recall that $\mathbf{I}^{\mathrm{BA}}$ denotes the set of the invariant measures for BA. Recall the definitions (14) and (15) of τ and of Θ_+, and recall that the dynamics of BA is denoted by A. It may be verified straightforwardly that

$$\text{if } \zeta \in \Theta_+ \text{ then } \tau(\zeta) = A(\zeta) \tag{16}$$

It follows then easily from (16) that if $\mu_1 \in \mathbf{P}^{\mathrm{BA}}_{+,\tau}$ then $\mu_1 \in \mathbf{I}^{\mathrm{BA}}$. Analogously, if $\mu_2 \in \mathbf{P}^{\mathrm{BA}}_{-,\tau}$ then $\mu_2 \in \mathbf{I}^{\mathrm{BA}}$. Since, according to the conditions of Theorem 1, μ_3 from (13) is picked from a subset of $\mathbf{I}^{\mathrm{BA}}$ and since, obviously, $\delta_\phi \in \mathbf{I}^{\mathrm{BA}}$ then μ defined by (13) must belong to $\mathbf{I}^{\mathrm{BA}}$. This completes the proof of the "if" part of the theorem. The "only if" part is more difficult. It will be proved with the aid of the following.

Lemma 3. Let $\zeta_0 \in \{-1, 0, 1\}^{\mathbb{Z}}$ and $m \in \mathbb{N}$ be arbitrarily fixed. Let ζ_m denote the configuration of particles in the BA at time m, starting from ζ_0, *i.e.*, $\zeta_m = A^m \zeta_0$. Take any pair of diverging particles in ζ_m (the term *diverging* has been defined in the beginning of the section). Then the distance between them is not less than $2m+1$.

Proof of Lemma 3. Find any pair of diverging particles in ζ_m. Observe that the initial position (i.e., in ζ_0) of the negative particle of this pair must be to the left of that of the positive one, because if not then these particle would have annihilated each other by time m. The assertion follows from this observation and from the fact that these particles were diverging with velocity 2 during the time interval $[0, m]$. ∎

Continuation of the Proof of Theorem 1(a). Lemma 3 provides that if $\mu \in \mathbf{I}^{\mathrm{BA}}$ then μ gives weight zero to any configuration that has at least one pair of diverging particles. Thus, such μ must be concentrated on $\Theta_+ \cup \Theta_- \cup \Theta_s \cup \{\phi\}$. Note however, that the dynamics A does not mix the sets Θ_+, $\Theta_- \Theta_s$ and $\{\phi\}$, that is, there is no a configuration ζ from one of these sets such that $A\zeta$ belongs to another one (note that it is the condition (ii) from (6) that guarantees that the set $A(\Theta_s)$ has no intersection with $\Theta_+ \cup \Theta_- \cup \{\phi\}$). Thus, the expansion (13) follows.

That the expansion (13) is unique follows from the fact that any two of the sets Θ_+, Θ_-, Θ_s, $\{\phi\}$ have empty intersection.

The statement that $\mathbf{P}^{\mathrm{BA}}_s$ does not contain translation invariant measures can be established by the following reasoning: Let $\mathrm{pos}(\zeta)$ denote the position of the positive particle of the unique pair of converging particles from $\zeta \in \Theta_s$. Observe that $\mathrm{pos}(\tau\zeta) = \mathrm{pos}(\zeta) + 1$ (τ is from (14)). Thus, if $\mu \in \mathbf{P}^{\mathrm{BA}}_s$ and if μ is translation invariant then $\mu\{\zeta \in \Theta_s : \mathrm{pos}(\zeta) = i\}$ acquires the same value for each $i \in \mathbb{Z}$. Since this is impossible, the statement is proved.

Proof of Theorem 1(b). Principal ideas of the proof are illustrated by Fig. 2.

Recall, from the text before the formulation of Theorem 1, the construction of mapping F_0 on set E'. We now state that this construction implies that for any $e \in E'$, it holds that: (a) among the rays that stem from all kinks of e, no two can intersect line $y=0$ at the same point; (b) the space coordinate of the intersection point of any ray with line $y=0$ is an integer number; (c) any kink creates a pair of particles such that the left-most (resp., right-most) of them is positive (resp., negative); (d) if $t_1 < t_2$ are two kink times then the particles created by kink $(e_{t(1)}, t_1)$ lie in the interval delimited by the particles created by kink $(e_{t(2)}, t_2)$; (e) the particles created by the same kink will annihilate each other in BA that starts from $F_0(e)$. The statements (a)–(d) follow via direct geometric analysis from the definition of F_0 and the property that $e_i - e_{i+\frac{1}{2}} = \pm\frac{1}{2}$, for all $i \in \frac{1}{2}\mathbb{Z}$. The statement (e) follows from (c) and (d). Combining statements (a)–(d) with the fact that $e \in E'$ has infinitely many kinks after time 0 (as the definition (9) assures), we conclude that $F_0(e) \in \Theta_s$ for any $e \in E'$.

For each $n \in \mathbb{Z}$, we define mapping F_n on set E' in the same way as F_0 has been, but for line $y=n$ in the place of $y=0$. By the argument applied above to F_0, we get that

$$F_n : E' \to \Theta_s \quad \forall n \in \mathbb{Z} \tag{17}$$

Our next objective is to establish the relation

$$F_{n+1}(e) = A(F_n(e)) \qquad \forall n \in \mathbb{Z} \text{ and } \forall e \in E' \tag{18}$$

which we prove below for $n=0$; for other n, the proof is analogous.

From the construction of F_0 and F_1, we get that a kink of e creates a pair of particles in $F_0(e)$ and does not create any particle in $F_1(e)$, if and only if the time coordinate of this kink is either 1/2 or 1. In any case, the distance between the pair of particles from $F_0(e)$ that are created by such a kink is not greater than 1, and thus, due to the property (e), they will have annihilated one another by time 1, and consequently, they are not present in $A(F_0(e))$. Consider now an arbitrary kink of e whose time-coordinate is larger than 1, and consider the pairs of particles in $F_0(e)$ and in $F_1(e)$ that are created by this kink. By the construction, the position of the positive (resp., negative) particle of the pair in $F_0(e)$ is by 1 to the right (resp., left) of the position of that in $F_1(e)$. Also, by the construction, the distance between particles in $F_0(e)$ is greater than 1. Combining these conclusions with (e), we get that the positions of the considered particles in $A(F_0(e))$ coincide with the positions of the considered particles in $F_1(e)$. Since we have proved (in (c) above) that each kink creates a unique pair of particles, and since there are no other particles rather than those created by kinks, then $F_1(e) = A(F_0(e))$ follows.

Using the mappings $F_n, n \in \mathbb{Z}$, we now define

$$F(e) := \big(\dots, F_{-1}(e), F_0(e), F_1(e), \dots\big) \quad \forall e \in E' \tag{19}$$

Eqs. (17) and (18) imply that the mapping F defined by (19) maps E' into $\Omega_s \subset \Theta_s^{\mathbb{Z}}$, where

$$\Omega_s := \{\omega = (\dots, \omega_{-1}, \omega_0, \omega_1, \dots) \equiv \{\omega_i\}_{i \in \mathbb{Z}} : \text{ both } \omega_n \in \Theta_s \text{ and } \omega_{n+1} = A\omega_n \ \forall n \in \mathbb{Z}\} \tag{20}$$

Note that, according to definitions, we can interpret a sequence ω from Ω_s as a BA process. This interpretation will be employed below in our constructions.

The next step in our proof is to construct mapping $G : \Omega_s \to E'$ and to show then that it is inverse to F.

Let $\omega \in \Omega_s$ be arbitrarily fixed. We transform the sequence $\omega = \{\omega_i\}_{i \in \mathbb{Z}}$ in the sequence $\hat{\omega} = \{\hat{\omega}_n\}_{n \in \frac{1}{2}\mathbb{Z}}$ in the following manner: for each $n \in \mathbb{Z}$, we define $\hat{\omega}_n := \omega_n$, and we define $\hat{\omega}_{n+\frac{1}{2}}$ as the configuration of A-particles in BA starting from ω_n after time 1/2 has passed. (Note that the interpretation of the BA dynamics given in Section 2.2 allows one to determine uniquely the particle that are present at any given time in BA, starting from any given configuration, as well as the positions and velocities of these particles.) According to our construction, we can interpret $\hat{\omega}$ as BA process in which particle positions and velocities are recorded at times from the set $\frac{1}{2}\mathbb{Z}$. This interpretation of $\hat{\omega}$ will be used below to define the second class particle.

We add to the process $\hat{\omega}$ a new particle called *the second class particle* whose position is determined by the rules *(i)–(iii)* specified below (see Figure 2 for an illustration). (i) It is an extra particle that does not affect the evolution of other particles and never disappears. (ii) It moves with velocity $+1$ or -1 along the same $\mathbb{R}$ on which the process' A-particles move. Its velocity may changes instantaneously according to the following rules: (ii-a) when, while going with velocity $+1$, the second class particle meets a negative particle it changes its velocity to -1 and starts to escort the met particle; (ii-b) at the time the escorted particle is annihilated, the second class particle changes its velocity to $+1$. (iii) If two particles annihilate at a time $n \in \frac{1}{2}\mathbb{Z}$ then at this time, the second class particle is at the annihilation point and changes its velocity from -1 to $+1$.

We observe that if $k \in \frac{1}{2}\mathbb{Z}$ is an annihilation time in $\hat{\omega}$ (by which we mean that two particles of $\hat{\omega}$ annihilate at time k) then we know (from the particle positions in $\hat{\omega}_{k-\frac{1}{2}}$) where this annihilation occurs, and consequently, rule (iii) allows us to define uniquely the second class particle position at time k in the process $\hat{\omega}$. But once the second class particle position in $\hat{\omega}$ is determined for some time k, the rule (ii) allows us to determine its position in $\hat{\omega}$ for any other time. Thus, $\hat{\omega}$ and any its annihilation time k determine uniquely the second class particle trajectory. We now want to prove that this trajectory will be the same, if we take another annihilation time. So, we take the same $\hat{\omega}$ and the annihilation time k' which is the immediate successor of k (i.e., no any annihilations in the time interval (k, k')). We consider two second class particle trajectories: one determined by $\hat{\omega}$ and k and another one determined by $\hat{\omega}$ and k'. It is easy to check that rules (iii) obliges the former trajectory to pass through the annihilation point at time k', and also obliges the latter to pass through the annihilation point at time k. It is also easy to check that then rule (ii) obliges them to coincide before time k, after time k', as well as within the time interval (k, k'). Thus, the second class particle trajectory in $\hat{\omega}$ is uniquely determined, once there is at least one annihilation in $\hat{\omega}$. But since $\hat{\omega}$ is constructed from $\omega \in \Omega_s$, then $\hat{\omega}$ possesses at least one (actually infinite) annihilation points. Consequently, we can state that the rules (i)–(iii) determine uniquely the second class particle trajectory from $\hat{\omega}$.

We denote by $e_i(\hat{\omega})$ the position of the second class particle in $\hat{\omega}$ at time $i \in \frac{1}{2}\mathbb{Z}$, and define then the mapping G on Ω_s via

$$G(\omega) := \{e_i(\hat{\omega})\}_{i \in \frac{1}{2}\mathbb{Z}} \quad \forall \omega \in \Omega_s \tag{21}$$

Remark 5. Note that in the definition of G, we do not specify the space from where $\hat{\omega}$ is picked, since it is sufficient for our needs that $\hat{\omega}$ is constructed from ω in a unique way. It is also not difficult to see that ω is uniquely determined by $\hat{\omega}$, so there is neither loss no gain of information in the passage from ω to $\hat{\omega}$. However, it is easier to work with $\hat{\omega}$ rather than with ω since in $\hat{\omega}$ particle annihilations occur at times when we observe its particle positions.

Let us prove that $e := G(\omega) \in E'$ for any $\omega \in \Omega_s$. To this end, first note that since the second class particles moves with velocity either $+1$ or -1, and since it may change the velocity solely at times from $\frac{1}{2}\mathbb{Z}$ (because an annihilation occurs solely at a time from $\frac{1}{2}\mathbb{Z}$), then $e_{n+\frac{1}{2}}(\hat{\omega}) - e_n(\hat{\omega}) = \pm\frac{1}{2}$, for any $n \in \frac{1}{2}\mathbb{Z}$, and consequently, $e \equiv G(\omega) \in E$. Now, since $\omega \in \Omega_s$, then $\omega_n \in \Theta_s$ for any time n, that is, at any time n there is pair of converging particles that will annihilate each other in a finite time. Thus, ω and, consequently, $\hat{\omega}$, has infinitely many annihilations after any time t. But since rules (i)–(iii) determine that the second class particle trajectory has a kink every time, when $\hat{\omega}$ has an annihilation, then $G(\omega) \in E'$.

In the following two paragraphs shall show that

$$F : E' \overset{\text{bijectively}}{\longrightarrow} \Omega_s, \quad G : \Omega_s \overset{\text{bijectively}}{\longrightarrow} E', \text{ and } G = F^{-1} \tag{22}$$

Take arbitrarily $e \in E'$ and construct BA $\omega := F(e)$. We note that if (e_t, t) is a kink in e then the particles created by this kink by the mapping F annihilate each other at time t at point e_t. Since each kink creates exactly one pair of particles then the sets of kinks of $G(\omega)$ and of e coincide. Due to the property (10), we have then that $G(F(e)) = e$.

Take now arbitrarily a ballistic annihilation process ω from Ω_s. By the construction of Ω_s, there is an annihilating companion to each particle from ω. Due to the definition of G, each pair of annihilating companions from ω creates a kink in $G(\omega)$, and there are no other kinks rather than those created by annihilations in ω. Take now an arbitrary pair of annihilating companions from ω and draw on the Euclidean plane (described after Eq. (8)) the "life line" of each particle from the pair, whereas the *life line* refers to the following set:

$$\begin{aligned}&\{(\text{the particle position at time } t,\ t)\\ &\quad \in \mathbb{R}^2 : t \in (-\infty,\ \text{the particle annihilation time}]\}\end{aligned} \tag{23}$$

Since any particle velocity equals 1 in modulus, then the drawn life lines emerge from the annihilation point in north-west and in north-east directions. Thus, these lines coincide with the rays that stem from this annihilation point and are used in the construction of the mapping F. Consequently, $F(G(\omega)) = \omega$.

The relations $G(F(e)) = e \forall e \in E'$ and $F(G(\omega)) = \omega \forall \omega \in \Omega_s$, just proved and the fact that E' and Ω_s are the domains of F and G, respectively, altogether imply (22).

The arguments employed above to derive (22) imply easily the following property, that says that F and G commute with time shifts:

$$F \circ \sigma\,(e) = \sigma \circ F\,(e) \quad \forall e \in E', \text{ and } \sigma \circ G(\omega) = G \circ \sigma(\omega) \quad \forall \omega \in \Omega_s \tag{24}$$

where σ in the l.h.s. of each equality shifts E' according to (11), while σ in the r.h.s. of each shifts Ω_s according to the follows definition:

$$(\sigma(\omega))_n := \omega_{n-1} \quad \forall n \in \mathbb{Z}, \quad \omega \in \Omega_s \tag{25}$$

Let us denote by $\mathbf{D}_\sigma$ the set of measures on Ω_s that are invariant with respect to σ defined in (25). Recall that $\mathbf{E}_\sigma$ consists of the measures on E' that are invariant with respect to σ defined in (11). For a measure $\varepsilon \in \mathbf{E}_\sigma$ let $F^*(\varepsilon)$ denote the measure on Ω_s such that $F^*(\varepsilon)(U) := \varepsilon\left(F^{-1}(U)\right) \equiv \varepsilon(G(U))$, for all $U \subset \Omega_s$. The definitions of $\mathbf{D}_\sigma$, $\mathbf{E}_\sigma$ and the relations (22) and (24) imply that

$$F^* : \mathbf{E}_\sigma \overset{\text{bijectively}}{\longrightarrow} \mathbf{D}_\sigma \tag{26}$$

Since Ω_s consists of (particular) trajectories of BA process, then each element D from $\mathbf{D}_\sigma$ may be interpreted as a distribution of a discrete time stochastic process, whose state space is $\{-1, 0, 1\}^{\mathbb{Z}}$. Using this interpretation, we define $H^*(D)$ as the marginal distribution at time 0 of D. Since, by the definition, D is invariant with respect to shift σ then all its marginals are identical. But then since for any $\omega \in \Omega_s$ the definitions (20) of Ω_s guarantees that $\omega_{n+1} = A\omega_n$, then $H^*(D) \in \mathbf{I}^{\mathrm{BA}}$, i.e., $H^*(D)$ is invariant for BA. On the other hand, the condition $\omega_n \in \Theta_s$ in the same definition implies that $H^*(D) \in \mathbf{P}_s^{\mathrm{BA}}$. Thus, H^* maps $\mathbf{D}_\sigma$ into $\mathbf{P}_s^{\mathrm{BA}} \cap \mathbf{I}^{\mathrm{BA}}$. Let us show that actually,

$$H^* : \mathbf{D}_\sigma \overset{\text{bijectively}}{\longrightarrow} \mathbf{P}_s^{\mathrm{BA}} \cap \mathbf{I}^{\mathrm{BA}} \tag{27}$$

To this end, let us pick an arbitrary $\mu \in \mathbf{P}_s^{\mathrm{BA}} \cap \mathbf{I}^{\mathrm{BA}}$ and construct BA process whose marginal distribution is μ at each time $n \in \mathbb{Z}$. Since $\mu \in \mathbf{I}^{\mathrm{BA}}$ then D, the distribution of the constructed BA, is invariant with respect to the shift σ. On the other hand, since $\mu \in \mathbf{P}_s^{\mathrm{BA}}$ then μ is concentrated on Θ_s and thus, D is concentrated on Ω_s. Consequently, $D \in \mathbf{D}_\sigma$. By the construction of D and H^*, we then have that $H^*(D) = \mu$. Thus, (27) is established.

The assertion (b) of Theorem 1 follows from (26), (27) just established and the following relation

$$H^*(F^*(\varepsilon)) = F_0^*(\varepsilon) \quad \forall \varepsilon \in \mathbf{E}_\sigma \tag{28}$$

This relation follows directly from our constructions. Indeed, F_0^* takes distribution ε on the second class particle trajectories and constructs from it $F_0^*(\varepsilon)$, the distribution of BA at time 0. On the other hand, F^* constructs the distribution on the trajectories of BA that corresponds to the distribution ε on the trajectories of the second class particle, and then, H^* extracts from $F^*(\epsilon)$ the distribution of BA at time 0. Thus, (28) is established and the theorem is proved. ∎

3.2. Characterization of Invariant Measures for CA 184

We note that although $\{0,1\}^{\mathbb{Z}}$ may be formally seen as a subset of $\{-1,0,1\}^{\mathbb{Z}}$, however, in our text, the former is interpreted as the state space for CA 184, while the latter as that for BA.

We shall say that an $\eta \in \{0,1\}^{\mathbb{Z}}$ has a *hole* at a site $x \in \mathbb{Z}$, if $\eta(x) = 0$. We then denote by *odd* and *even* two configurations from $\{0,1\}^{\mathbb{Z}}$ such that *odd* (resp., *even*) has a particle (resp., a hole) at each odd $i \in \mathbb{Z}$, and has a hole (resp., a particle) at each even $i \in \mathbb{Z}$. Using the term "consecutive" defined in Section 3.1, we construct

$$\begin{aligned}\Lambda := \{\zeta \in \{-1,0,1\}^{\mathbb{Z}} :& \\ &\text{the distance between any consecutive particles} \\ &\text{with the same velocity is odd and} \\ &\text{the distance between any consecutive particles} \\ &\text{with opposite velocities is even}\}\end{aligned} \tag{29}$$

We then state that $T \equiv T_{184,BA}$, that has been defined in (4), satisfies the following properties:

$$T : \{0,1\}^{\mathbb{Z}} \setminus \{\text{odd, even}\} \to \Lambda \setminus \{\phi\} \subset \{-1,0,1\}^{\mathbb{Z}} \text{ is a bijection} \tag{30}$$

$$T(\text{odd}) = T(\text{even}) = \phi \tag{31}$$

where ϕ denotes the empty configuration from $\{-1,0,1\}^{\mathbb{Z}}$. Properties (30) and (31) follow straightforwardly and easily from definitions, nevertheless, they play the central role in the derivation of Theorem 2 from Theorem 1.

For a measure μ supported by $\Lambda \setminus \{\phi\}$, let $T^*(\mu)$ denote the measure on $\{0,1\}^{\mathbb{Z}}$ such that $T^*\mu(U) := \mu(TU)$ for any $U \subseteq \{0,1\}^{\mathbb{Z}} \setminus \{\text{odd, even}\}$. The properties (30) and (31) imply then that

$$\begin{aligned}T^* : \{\text{measures supported by } \Lambda \setminus \{\phi\}\} &\overset{\text{bijectively}}{\longrightarrow} \\ \{\text{measures supported by } \{0,1\}^{\mathbb{Z}} \setminus &\{\text{odd, even}\}\}\end{aligned} \tag{32}$$

Let us construct (below, E' is from (9) and $\mathbf{E}_\sigma$ has been defined after (11))

$$\begin{aligned}\tilde{E}' := \{\, e \in E' : e \text{ cannot have kinks at non-integer times, i.e.,} & \\ e_{j-\frac{1}{2}} - e_j = e_{j+\frac{1}{2}} - e_j = 1 \text{ may occur only if } j \in \mathbb{Z}\}&\end{aligned} \tag{33}$$

$$\mathbf{E}_\sigma^{184} := \{\, \nu \in \mathbf{E}_\sigma : \nu \text{ is supported by } \tilde{E}'\} \tag{34}$$

We shall show in the proof of Theorem 2 that (F_0^* used below was defined in (12))

$$F_0^* : \mathbf{E}_\sigma^{184} \overset{\text{bijectively}}{\longrightarrow} \{\mu \in \mathbf{P}_s^{\mathrm{BA}} \cap \mathbf{I}^{\mathrm{BA}} : \mu \text{ is supported by } \Lambda \setminus \{\phi\}\} \tag{35}$$

Properties (32) and (35) guarantee that $T^* \circ F_0^*$ is a well defined mapping on the measure set $\mathbf{E}_\sigma^{184}$. The role of $T^* \circ F_0^*$ in characterization of the invariant measures for CA 184 is the same as that of F_0^* for BA (see Remark 2 and the text right before the formulation of Theorem 1).

Theorem 2. (**Characterization of invariant measures for CA 184**). Let δ_{odd} and δ_{even} denote the measures concentrated on the configuration *odd* and *even* respectively. Let $\mathbf{I}^{184}$ denote the set of the measures that are invariant for CA184.

(a) A measure λ on $\{0, 1\}^{\mathbb{Z}}$ is invariant for CA 184 if and only if

$$\lambda = \alpha_1\lambda_1 + \alpha_2\lambda_2 + \alpha_3\lambda_3 + \alpha_4\left(\frac{\delta_{\mathrm{odd}} + \delta_{\mathrm{even}}}{2}\right) \tag{36}$$

for some non-negative numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, such that $\alpha_1 + \cdots + \alpha_4 = 1$, and for some measures $\lambda_1, \lambda_2, \lambda_3$ from the respective sets $\mathbf{P}^{184}_{\tau,\text{particle blocks}}$, $\mathbf{P}^{184}_{\tau,\text{hole blocks}}$, $\mathbf{P}_s^{184} \cap \mathbf{I}^{184}$ that are described below. The expansion (36) is unique.

(b) $\mathbf{P}^{184}_{\tau,\text{particle blocks}}$ and $\mathbf{P}^{184}_{\tau,\text{hole blocks}}$ are the sets of the measures that are supported by, respectively, the configuration sets

$$\{\eta \in \{0, 1\}^{\mathbb{Z}} : \text{ no two adjacent sites of } \mathbb{Z} \text{ are empty in } \eta\} \setminus \{\text{odd}, \text{even}\} \tag{37}$$

$$\{\eta \in \{0, 1\}^{\mathbb{Z}} : \text{ no two adjacent sites of } \mathbb{Z} \text{ have particles in } \eta\} \setminus \{\text{odd}, \text{even}\} \tag{38}$$

and that are translation invariant, i.e., are invariant with respect to the transformation

$$\tau : \{0, 1\}^{\mathbb{Z}} \to \{0, 1\}^{\mathbb{Z}} \text{ defined by } (\tau(\eta))(i) = \eta(i-1) \quad \forall i \in \mathbb{Z}\ \forall \eta \in \{0, 1\}^{\mathbb{Z}} \tag{39}$$

Informally speaking, a configuration from the set (37) (resp., (38)) consists of blocks of particles (resp., holes) that are separated by sole holes (resp., particles).

(c) $\mathbf{P}_s^{184}$ consists of the measures that are concentrated on the following configuration set:

$$\big\{\eta \in \{0,1\}^{\mathbb{Z}} : \text{exists}\, x = x(\eta) \in \mathbb{R} \text{ such that no two adjacent sites of } \mathbb{Z} \text{ to the left of}\, x \text{ contain particles, while no two adjacent sites of } \mathbb{Z} \text{ to the right of}\, x \text{ are free of particles}\big\}$$

The set $\mathbf{P}_s^{184}$ does not contain translation invariant measures. The measures from $\mathbf{P}_s^{184}$ that are invariant for CA 184 form the set, denoted by $\mathbf{P}_s^{184} \cap \mathbf{I}^{184}$, which is isomorphic to the measure set $\mathbf{E}_\sigma^{184}$ by the mapping $T^* \circ F_0^*$, whereas T^*, F_0^* and $\mathbf{E}_\sigma^{184}$ are from (32), (12) and (34) respectively.

Proof. Recall from (6) the construction of the configuration set Θ_s and note that $\phi \notin \Theta_s$, so that we can write $\Theta_s \cap \Lambda$ in the place of its equivalent $\Theta_s \cap \{\Lambda \setminus \{\phi\}\}$. Let us define

$$\tilde{\Omega}_s := \{\omega \in \Omega_s : \omega_n \in \Theta_s \cap \Lambda \quad \forall n \in \mathbb{Z}\} \subset \Omega_s \tag{40}$$

Recall from the proof of Theorem 1 the construction of maps F and G. It may be checked straightforwardly that

$$F : \tilde{E}' \overset{\text{bijectively}}{\longrightarrow} \tilde{\Omega}_s, \quad G : \tilde{\Omega}_s \overset{\text{bijectively}}{\longrightarrow} \tilde{E}', \quad \text{and} \quad G = F^{-1} \quad \text{on } \tilde{\Omega}_s \tag{41}$$

Relations (41) imply (35) by the same argument as that used in the proof of Theorem 1 from (22) that F_0^* maps bijectively $\mathbf{E}_\sigma$ onto $\mathbf{P}_s^{\mathrm{BA}} \cap \mathbf{I}^{\mathrm{BA}}$.

From Lemma 1, from (32) and the definitions of the measure sets involved, it follows directly that

$$\begin{aligned} T^* \text{ maps bijectively } &\{\mu \in \mathbf{P}_{+,\tau}^{\mathrm{BA}} : \mu \text{ supported by } \Lambda\} \text{ onto } \mathbf{P}_{\tau,\text{hole blocks}}^{184}, \\ &\{\mu \in \mathbf{P}_{-,\tau}^{\mathrm{BA}} : \mu \text{ supported by } \Lambda\} \text{ onto } \mathbf{P}_{\tau,\text{particle blocks}}^{184}, \\ &\{\mu \in \mathbf{P}_s^{\mathrm{BA}} \cap \mathbf{I}^{\mathrm{BA}} : \mu \text{ is supported by } \Lambda\} \text{ onto } \mathbf{P}_s^{184} \cap \mathbf{I}^{184} \end{aligned} \tag{42}$$

(note that neither of the sets $\mathbf{P}_{+,\tau}^{\mathrm{BA}}$, $\mathbf{P}_{-,\tau}^{\mathrm{BA}}$, $\mathbf{P}_s^{\mathrm{BA}} \cap \mathbf{I}^{\mathrm{BA}}$ contains the measure δ_ϕ, so that we need not to subtract ϕ from Λ). The last line in (42) and (35), that has been justified in the above paragraph, imply the isomorphism between $\mathbf{E}_\sigma^{184}$ and $\mathbf{P}_s^{184} \cap \mathbf{I}^{184}$ as stated in (c) of Theorem 2. The proof that $\mathbf{P}_s^{184}$ does not contain translation invariant measures is similar to the proof that $\mathbf{P}_s^{\mathrm{BA}}$ does not.

Recall that the dynamics of CA 184 has been denoted by C. Note that $C(\text{odd}) = \text{even}$ and $C(\text{even}) = \text{odd}$. Thus, a measure λ is invariant for CA 184 if and only if it is represented uniquely as follows:

$$\lambda = \gamma\lambda' + (1-\gamma)\lambda'' \tag{43}$$

for some $\gamma \in [0,1]$, some measure λ' supported by $\{0,1\}^{\mathbb{Z}} \setminus \{\text{odd}, \text{even}\}$ and some measure λ'' supported by $\{\text{odd}, \text{even}\}$, whereas both λ' and λ'' are invariant for CA 184.

As for λ'', there is the unique choice for it, namely, $\lambda'' = (\delta_{\text{odd}} + \delta_{\text{even}})/2$; this fact is implied by the relations $C(\text{odd}) = \text{even}$, $C(\text{even}) = \text{odd}$.

Let us analyze λ'. Since $\lambda'(\{\text{odd}, \text{even}\}) = 0$, then $\mu := (T^*)^{-1}(\lambda')$ is a well defined measure, that is supported by $\Lambda \setminus \{\phi\}$ (due to (32)) and is invariant for BA (due to Lemma 1 and the use of T in the construction of T^*). Thus, according to Theorem 1, μ admits the expansion (13), in which $\alpha_4 = 0$, and each one of the measures μ_1, μ_2, μ_3 is supported by $\Lambda \setminus \{\phi\}$. Applying then (42), we get the expansion $\lambda' = \alpha_1 T^*(\mu_1) + \alpha_2 T^*(\mu_2) + \alpha_3 T^*(\mu_3)$. Since (13) is unique and T^* acts bijectively then the expansion for λ' is unique. This and (43) imply the "only if" part of the theorem.

To prove the "if" part, assume (36) holds. From (42) and Lemma 3, the measures λ_1 and λ_2 are invariant for CA 184. The measure λ_3 is invariant for CA 184, since it is picked from $\mathbf{P}_s^{184} \cap \mathbf{I}^{184}$. The invariance for CA 184 of $(\delta_{\text{odd}} + \delta_{\text{even}})/2$ follows straightforwardly. Thus, λ is invariant for CA 184. ∎

Remark 6. Recall that CA 184 has two "stochastic counterparts" TASEP and CA&N that we described briefly in Section 2.1. Refs. 18 and 23 showed that the set of the non-translation invariant measures that are invariant for TASEP and CA&N is $\{\nu^{(n)}, -\infty < n < \infty\}$, where $\nu^{(n)}$ gives mass 1 to the configuration $\eta^{(n)}$ such that $\eta^{(n)}(x) = 1\, \forall x \geqslant n$ and $\eta^{(n)}(x) = 0\, \forall x < n$. We thus, remark that $\mathbf{P}_s^{184} \cap \mathbf{I}^{184}$ is wider than this set.

4. HYDRODYNAMIC LIMIT

In this and the following sections, we shall consider processes indexed by the time set $\mathbb{N}$ rather than $\mathbb{Z}$, since the questions to be addressed concern the long time behavior of processes starting from particular distributions, and require, thus, the "starting time" would be settled down; it is $n = 0$.

Let, as usual, $\mathcal{C}[a,b]$ denote the space of continuous real valued functions on $[a,b]$, and let $\mathcal{C} := \mathcal{C}(-\infty, +\infty)$. Let then

$$\tilde{\mathcal{C}} := \{ f \in \mathcal{C} : f(n) \in \mathbb{Z} \, \forall n \in \mathbb{Z}, \\ \text{and } f \text{ is linear between any two adjacent integer abscissae} \} \tag{44}$$

We say that $f \in \tilde{\mathcal{C}}$ is an *integrated profile* for particle configuration $\zeta \in \{-1, 0, 1\}^{\mathbb{Z}}$ (i.e., for BA particle configuration), if

$$f(n) - f(n-1) = \zeta(n) \quad \forall n \in \mathbb{Z} \tag{45}$$

and we say that $f \in \mathcal{R}$ (the space $\mathcal{R}$ has been defined in Section 2.3) is an *integrated profile* for particle configuration $\eta \in \{0, 1\}^{\mathbb{Z}}$ (i.e., for CA 184 particle configuration), if (compare to (5))

$$(1 - f(n) + f(n-1))/2 = \eta(n) \quad \forall n \in \mathbb{Z} \tag{46}$$

Certainly, there is no a loss of information in representing a particle configuration by an integrated profile, while a gain is that it allows one to formalize and treat the problem of the following nature: if in a BA or CA 184 process $\{\zeta_n, n \in \mathbb{N}\}$ the particle positions at time n are re-scaled by some factor a_n^{-1}, would the re-scaled particle distributions converge to some limit and what would be its properties? For this question to make sense, it is necessary that all the re-scaled profiles be put in the same space. The easiest way to achieve this is to represent them by integrated profiles, since all those are from the same space $\mathcal{C}$. In terms of integrated profiles, this problem reads: Do there exist, for every $n \in \mathbb{N}$, two real numbers c_n and a_n and a function $f_n(\cdot)$ such that $f_n(\cdot)$ is an integrated profile for ζ_n and $c_n f_n(a_n \cdot)$ converges to a limit, as $n \to \infty$? The limit in question is called *hydrodynamic limit* (a general treatment of hydrodynamic limits for particle systems is given in book 14). It is described in Theorems 3 and 4 for respectively, BA and CA 184.

Note that for any particle configurations, all its integrated profiles form a class such that any function from this class may be obtained from another one by a vertical shift. As far as particle positions are concerned, any function from the corresponding class bears the same information, and thus, it is natural that the question that concerns hydrodynamic limit does not impose conditions on exactly which representative from each class should be chosen as $f_n(\cdot)$. To pick the "right" representative is as important for a success in establishing hydrodynamic limit as to find the correct scaling factors c_n and a_n. Also, it is important for the success that integrated profile be appropriately constructed. Note in respect, that there are infinite ways to present a particle configuration by a profile; for example, the rule (45) might be an alternative to (46) for constructing integrated profile for CA 184 particles. With (45) in the place of (46), however, the counterpart of Theorem 4 would not follow from Theorem 3 as easily and cleanly as it does.

We draw the reader attention to the fact that the process' convergence discussed everywhere below is understood as the weak convergence of process' distributions on $\mathcal{C}[a,b]$ for any $-\infty<a<b<+\infty$ (see ref. 3).

Theorem 3. For $y\geqslant 0$, let M_y denote the operator that brings a function $f(\cdot):\mathbb{R}\to\mathbb{R}$ to the function $g(\cdot):\mathbb{R}\to\mathbb{R}$ in the way such that

$$g(x)=\big(M_y f\big)(x)=\min\{f(z)\,:\,x-y\leqslant z\leqslant x+y\}\qquad \forall x\in\mathbb{R} \tag{47}$$

(a) If a random function $f_0(\cdot)\in\tilde{\mathcal{C}}$ is an integrated profile, in the sense of (45), of a random configuration $\zeta_0\in\{-1,0,1\}^{\mathbb{Z}}$ then, for any $n\in\mathbb{N}$, the function

$$f_n(\cdot):=\big(M_1^n f_0\big)(\cdot)=(M_n f_0)(\cdot) \tag{48}$$

is an integrated profile, in the sense of (45), for $\zeta_n:=A^n\zeta_0$, the particle configuration at time n in BA that starts from ζ_0.

(b) (Hydrodynamic limit for BA.) Let ζ_0 and $f_0(\cdot)$ be as in item (a) and suppose that there exist a sequence of real numbers $\{c_n, n\in\mathbb{N}\}$ and a stochastic process $W(t), t\in\mathbb{R}$, whose trajectories belong to $\mathcal{C}$, such that

$$c_n f_0(n\cdot)\to W(\cdot)\qquad \text{as } n\to\infty \tag{49}$$

then Eq. (48) defines the sequence $\{f_n(\cdot), n\in\mathbb{N}\}$ of integrated profiles of BA $\{A^n\zeta_0, n\in\mathbb{N}\}$ that satisfies the following hydrodynamic limit relation:

$$c_n f_n(n\cdot)\to (M_1 W)(\cdot)\qquad \text{as } n\to\infty \tag{50}$$

Note that the process $(M_1 W)(\cdot)$ may be called *the moving local minimum* of the process $W(\cdot)$, since its value at each "time" t is the minimum of the values of $W(\cdot)$ in the "time window" of length 1 centered at t.

Theorem 3 has been established in ref. 1 for a particular case when ζ_0, a_n and c_n are such that the process $W(\cdot)$ is the standard Brownian motion (naturally, ζ_0 is a Bernoulli product measure on $\{-1,+1\}^{\mathbb{Z}}$, and $c_n=n^{-1/2}$,

$a_n = n$). The objective of [1] was to establish the distribution of the moving local minimum of the Brownian motion. To this end, the distribution of $f_n(\cdot)$ was calculated and then, the limit distribution of $n^{-1/2} f_n(n\cdot)$ was found. According to (50), the found law is the desired law of the moving local minimum of Brownian motion. Thus, the work[1] illustrates that Theorem 3 may be applied to derive a process distribution via considering an appropriate BA or CA 184 process. Note that this theorem may be also used in the "opposite" direction: if one knows that (49) holds for some known process $W(\cdot)$ and numeric sequence c_n, then one may use $(M_1 W)(\cdot)$ and (50) in order to get an approximate shape of $f_n(\cdot)$, and consequently, an approximate *charge* of particles in BA at time n in any interval $[a, b]$ (via the obvious relation: $f_n(b) - f_n(a-1) = \sum_{i=a}^{b} \zeta_n(i) =$ #(positive particles) - #(negative particles) at time n in $[a, b]$).

Proof of Theorem 3. Let $\zeta_0 \in \{-1, 0, +1\}^{\mathbb{Z}}$ be arbitrary and let $f_0(\cdot)$ be its integrated profile (any one from the class of the integrated profiles of ζ_0). The fact that $(M_1 f_0)(\cdot)$ is an integrated profile for the configuration $A\zeta_0$ follows by a straightforward verification (see Fig. 1 for an illustration). To complete the proof of (a) we thus, only have to establish that $M_1^n = M_n$ for any n. But this fact follows directly from the definition (47) of M_y.

Let us now prove (b). First, we note that for any function $h: \mathbb{R} \to \mathbb{R}$, the modulus of continuity (which definition one finds in ref. 3) of $M_1 h$ does not exceed that of h; this property of M_1 can be verified straightforwardly. This fact allows us to apply Theorem 5.1 from ref. 3 to the assumption $c_n f_0(n\cdot) \to W(\cdot)$ to conclude that

$$(M_1\big(c_n f_0(n\cdot)\big))(\cdot) \to \big(M_1 W\big)(\cdot) \tag{51}$$

We must comment on the use of two "$\cdot$" in (51) in order to avoid a possible confusion in its interpretation: $(M_1(c_n f_0(n\cdot)))(\cdot)$ means the function obtained by rescaling f_0 by c_n along the axis of ordinates and by n^{-1} along the axis of abscissa, and then, by applying M_1. It may be verified directly that for any continuous function h and any constant c,

$$(M_1(ch(n\cdot)))(x) = c\big(M_n h\big)(nx) \qquad \forall x \in \mathbb{R}$$

Thus, the l.h.s. of (51) is equal to $c_n(M_n f_0)(n\cdot)$, which in turn, is equal to $c_n f_n(n\cdot)$, according to the choice of f_n. Thus, (50) follows from (51), and the theorem is proved. ∎

Theorem 4. (**Hydrodynamic limit for CA 184**). Assume that a random function $g_0(\cdot)\in\mathcal{R}$ is an integrated profile, in the sense of (46), of a random configuration $\eta_0\in\{0,1\}^{\mathbb{Z}}$, and that there exist a sequence of real numbers $\{c_n, n\in\mathbb{N}\}$ and a stochastic process $W(t), t\in\mathbb{R}$, whose trajectories belong to $\mathcal{C}$, such that

$$c_n\to 0 \text{ as } n\to\infty \text{ (see Remark 7 for the role of this assumption)} \tag{52}$$

$$c_n g_0(n\cdot)\to W(\cdot) \text{ as } n\to\infty \tag{53}$$

For each $n\in\mathbb{N}$, let $\eta_n := C^n\eta_0$ denote the particle configuration in CA 184 at time n, starting from η_0, and let $g_n(\cdot)$ be the integrated profile, in the sense of (46), of η_n such that

$$g_n(0)=M_n g_0(0) \tag{54}$$

Then,

$$c_n g_n(n\cdot)\to M_1 W(\cdot) \text{ as } n\to\infty \tag{55}$$

Remark 7. The cornerstone of the proof is the property (56) that says that integrated profiles of a BA particle configuration and a CA 184 particle configuration go hand in hand, if they have a common point and if the configurations are related via $T_{184,\mathrm{BA}}$ from (4). Conditions (52) and (54) assure then that BA and CA 184 must have the same hydrodynamic limit, if their initial states are related via $T_{184,\mathrm{BA}}$.

Proof. Let $T := T_{184,\mathrm{BA}}$ be the mapping defined in (4), let $\zeta\in\{-1,0,1\}^{\mathbb{Z}}$ and $\eta\in\{0,1\}^{\mathbb{Z}}$ be arbitrary but such that $T\eta=\zeta$, and let $f(\cdot)$ and $g(\cdot)$ be integrated profiles of ζ and η, respectively (in the sense of (45) and (46), respectively). It may be verified straightforwardly (Fig. 1 might be to hand here) that

$$\text{if } g(0)=f(0) \text{ then } |f(x)-g(x)|\leqslant 1 \quad \forall x\in\mathbb{R} \tag{56}$$

We introduce $\zeta_0 := T\eta_0$ and we denote by $f_0(\cdot)$ the integrated profile of ζ_0 such that $f_0(0)=g_0(0)$. The latter equality and the relations (56), (52) and (53) imply the relation (49), which due to Theorem 3, implies that $c_n f_n(n\cdot)\to(M_1W)(\cdot)$, if $f_n(\cdot)$ is defined by (48) for each $n\in\mathbb{N}$. The latter convergence and the relations (56) and (52) would imply the theorem statement (55), if we had that

$$\text{for some constant } c', \ |g_n(0) - f_n(0)| \leqslant c' \text{ uniformly in } n \tag{57}$$

There are various conditions, not necessarily equivalent among themselves, that imply (57). One of them is the condition (54). Indeed, from our construction of $g_0(\cdot)$ and $f_0(\cdot)$ and from (56), we can conclude that $|M_n g_0(x) - M_n f_0(x)| \leqslant 1$, $\forall x$. On the other hand, $|g_n(x) - f_n(x)| \leqslant |g_n(0) - f_n(0)| + 1$, $\forall x$, because of (56), which applies to the case, since $\zeta_n \equiv A^n \zeta_0 = T\eta_n \equiv T(C^n \eta_0)$, as Lemma 1 assures. Combining the obtained inequalities with the condition (54) and the identity $M_n f_0 = f_n$, we get (57). Thus, the proof is completed. ▮

Theorem 5. **(Limit shape of functions in SG under a hydrodynamic scaling)**. Let $h_0(\cdot) \in \mathcal{R}$ be a random function, and $\{c_n, n \in \mathbb{N}\}$ be a sequence of real numbers such that

$$c_n \to 0 \quad \text{as } n \to \infty \tag{58}$$

$$c_n h_0(n\cdot) \to W(\cdot) \quad \text{as } n \to \infty \tag{59}$$

for some stochastic process $W(t), t \in \mathbb{R}$, whose trajectories belong to $\mathcal{C}$. Let $\{h_n(\cdot), n \in \mathbb{N}\}$ be SG process starting from $h_0(\cdot)$. Then, there exists a sequence of random variables $\{\alpha_n, n \in \mathbb{N}\}$ such that

$$c_n(h_n(n\cdot) + \alpha_n) \to M_1 W(\cdot) \quad \text{as } n \to \infty \tag{60}$$

Proof. We construct $\eta_0 \in \{0, 1\}^{\mathbb{Z}}$ via $\eta_0(n) := (1 - h_0(n) + h_0(n-1))/2 \forall n \in \mathbb{Z}$. By (46), h_0 is an integrated profile for η_0, and thus, putting $g_0 := h_0$, we conclude that the conditions (52) and (53) of Theorem 4 are satisfied, and consequently, $c_n g_n(n\cdot) \to M_1 W(\cdot)$, where $g_n(\cdot)$ is an appropriate integrated profile of $\eta_n := C^n \eta_0$. But Lemma 2 implies that $h_n(\cdot)$ is also an integrated profile for η_n, so that $g_n(\cdot) - h_n(\cdot) = \alpha_n$ for an appropriate random variable α_n. Thus, (60) follows. ▮

5. CONVERGENCE TO EQUILIBRIUM

5.1. The Domains of Attraction of Invariant Measures

The domains of attraction of invariant measures of BA are analyzed in Theorem 6 below. Its counterparts for CA 184 and SG may be derived in the manner analogous to the derivation of Theorem 2 from Theorem 1, employing the relations between the processes that have been established in Lemmas 1 and 2.

Theorem 6. **(Domain of attraction for invariant measures of BA).**

(a) Let μ be any translation invariant measure on $\{-1, 0, +1\}^{\mathbb{Z}}$. Then BA, starting from μ, converges to some measure from the set $\mathbf{P}^{\mathrm{BA}}_{-,\tau} \cup \mathbf{P}^{\mathrm{BA}}_{+,\tau} \cup \delta_{\phi}$.

(b) There exist measures from $\mathbf{P}^{\mathrm{BA}}_{\mathrm{s}}$ such that BA starting from any one of them, does not converge to any measure on $\{-1, 0, +1\}^{\mathbb{Z}}$.

Proof of Theorem 6(a). Let ζ be an arbitrary configuration of A-particles. Let us mark "condemned" each its particle that will be annihilated in BA staring from ζ. This may be achieved by repeating the following procedure: mark all converging pairs in ζ and remove them, mark all converging pairs of the remained particles and remove them, and so on. Let $\bar{\zeta}$ denote the configuration obtained from ζ by eliminating all its condemned particles. According to this procedure, $\bar{\zeta} \in \Theta_+ \cup \Theta_- \cup \{\phi\} \cup \Theta_{\text{diverging}}$, where Θ_+, Θ_- and ϕ have been defined in the formulation of Theorem 1 while

$$\Theta_{\text{diverging}} := \{\zeta \in \{-1, 0, 1\}^{\mathbb{Z}} : \text{ exists } x \in \mathbb{R} \text{ such that all positive (resp., negative) particles are situated to the right (resp., left) of } x\} \tag{61}$$

Let us denote by $\bar{\mu}$ the measure obtained from μ via the transformation $\zeta \to \bar{\zeta}$ of each ζ from the support of μ, namely, for each cylinder set $U \subset \{-1, 0, +1\}^{\mathbb{Z}}$, we set $\bar{\mu}(U) := \mu(\zeta : \bar{\zeta} \in U)$. Note that $\bar{\mu}$ is translation invariant, since both μ and the dynamics of BA were. On the other hand, as we have just established, $\bar{\mu}$ must be supported by $\Theta_+ \cup \Theta_- \cup \{\phi\} \cup \Theta_{\text{diverging}}$. However, employing the argument that has been used in the proof of Theorem 1 to show that $\mathbf{P}^{\mathrm{BA}}_{s}$ does not contain translation invariant measures, it is easy to derive that a translation invariant measure cannot be supported by $\Theta_{\text{diverging}}$. Thus $\bar{\mu}$ is supported by $\Theta_+ \cup \Theta_- \cup \{\phi\}$. Now, since the dynamics of BA does not mix these sets (as we have shown in the "Continuation of the proof of Theorem 1(a)") then $\bar{\mu}$ is a convex combination of three measures, all translation invariant, that are supported by respectively, Θ_+, Θ_- and $\{\phi\}$. Thus, Theorem 1 implies that $\bar{\mu}$ is invariant for BA.

Let $(A^*)^n\mu$ denote the distribution at time n of BA when the initial distribution is μ. We want to show that $(A^*)^n\mu \to \bar{\mu}$, as $n \to \infty$, which is equivalent to showing that $(A^*)^n\mu \to (A^*)^n\bar{\mu}$, as $n \to \infty$, since $\bar{\mu}$ is invariant for BA. To establish the desired result, we fix arbitrarily a finite region $\{i, i+1, \ldots, j\}$, $i, j \in \mathbb{Z}$, and consider the variational distance between the distributions of particles in this region that are determined by $(A^*)^n\mu$ and $(A^*)^n\bar{\mu}$. Since for any ζ and any n, the configuration of non-condemned particles of $A^n\zeta$ coincides with the configuration $A^n\bar{\zeta}$, then the considered variational distance is equal to

$$\mu\big[\text{there is at least one condemned particle of } \zeta \text{ in } \{i, i+1, \dots, j\} \text{ at time } n\big] \quad (62)$$

But if a particle of $A^n\zeta$ is in $\{i, i+1, \dots, j\}$ then its position in ζ belongs to either $\{i-n, i+1-n, \dots, j-n\}$ or to $\{i+n, i+1+n, \dots, j+n\}$. Thus, since μ is translation invariant, then (62) is not larger than $2(1 - h^{\mu}_{j-i}(n))$, where

$$h^{\mu}_k(n) := \mu\big[\text{all the condemned particles of } \zeta \text{ in the region } \{\ell+1, \dots, \ell+k\} \text{ will have died by time } n \text{ in BA, starting from } \zeta\big], \ell \in \mathbb{Z}, k \in \mathbb{N}, n \in \mathbb{N} \quad (63)$$

($h^{\mu}_k(n)$ does not depend on ℓ since μ is translation invariant). However, since each condemned particle dies sooner or later then

$$h^{\mu}_k(n) \nearrow 1 \text{ as } n \to \infty \text{ for any fixed } k \quad (64)$$

Thus, (62) decreases to 0 as $n \to \infty$, which in turn, implies the convergence of $(A^*)^n\mu$ to $\bar{\mu}$. This completes the proof of item (a).

Proof of Theorem 6(b). It is enough to present a configuration $\zeta \in \Theta_s$ such that BA, starting from ζ, does not converge to any measure. We shall construct such ζ with aid of the second class particle defined in the proof of Theorem 1.

Let the second class particle be at site 0 at time 0 and let its evolution consist of excursions from 0 and oscillations around 0, in alternating order. At the ith excursion, the second class particle starts from 0, moves to the site -2^i and comes back to 0; thus, the ith excursion duration is 2^{i+1}. After the second class particle comes back to 0, at the end of ith excursion, it oscillates between 0 and 1/2 during $2^{i+1}+1$ time units. Denote by e the trajectory of the second class particle. Recall the definition of operator F_0 from the text right before the formulation of Theorem 1. Let $\zeta := F_0(e)$. From the definition, $\zeta \in \Theta_s$. We now state that BA, starting from ζ, does not converge to any measure. Indeed, if it were then the second class particle position would have a limiting distribution. But our construction of e guarantees that such a distribution does not exist. ∎

5.2. Rate of Convergence

A BA process, starting from μ, is said to converge to its invariant state with rate $f(n)$ as "time" $n \to \infty$, if the variational distance between its particle distribution at time n and its limit particle distribution on any interval $[i, j] \subset \mathbb{Z}$ may be limited by $c_{i,j} f(n)$ from below and by $c'_{i,j} f(n)$ from above, for some constants $c_{i,j}, c'_{i,j}$ that do not depend on n (clearly, $f(n)$ here must not depend on i, j); certainly, actual values of the constants and the function may depend on the structure of μ. Usually, a constant factor of $f(n)$ is of no interest, so that typically a rate of convergence is presented as, for example, $n^{-1/2}$ or $(\log n)^{-1}$.

As we have shown in the proof of Theorem 6 (a), if μ is translation invariant, then an upper bound for the variational distance is $2(1 - h^{\mu}_{j-i}(n))$. It is straightforward to see that $1/2(1 - h^{\mu}_{j-i}(n))$ may be taken as its lower bound. Thus, we get the following general result: if $1 - h^{\mu}_{j-i}(n)$ behaves asymptotically (in n) as $c^{\mu}_{i,j} f^{\mu}(n)$ then $f(n)$ is the rate of convergence to the equilibrium measure of BA that starts from a translation invariant measure μ. Two examples below present particular situations when $c_{i,j} f(n)$ may be found relatively easily.

Example 1. Let μ be Bernoulli 1/2 on $\{-1, +1\}^{\mathbb{Z}}$. Let $p^{\mu}(n)$ denote the probability that the particle from 0 has not been annihilated by time n. Obviously, $p^{\mu}(n) \leqslant 1 - h^{\mu}_{k}(n) \leqslant k p^{\mu}(n)$. Let us estimate the rate of decay of $p^{\mu}(n)$ to 0. For this, we shall need two facts that may be verified straightforwardly. The first fact says that for two particles to annihilate each other by time n, it is necessary that they are at most $2n$ apart one from another at time 0. The second fact says that if m and m' $(m < m')$ are the positions of a positive particle and its annihilation companion in a configuration ζ, then the integrated profile (in the sense of (45)) of ζ attains the same value at abscissas $m - 1$ and m' and does not attain this value at any other point from $(m-1, m')$. (This fact is illustrated on Fig. 1(a): the annihilation companion of the $+$-particle at site 1 is the $-$-particle at site 11; accordingly, the integrated profile passes through 0 at 0 and returns to 0 for the first time, at 11.) But if the A-particles are distributed by μ then their integrated profile is a simple symmetric (one-dimensional) random walk. Using then classical results for this random walk (see ref. 11, Ch. 3), we get that, $p^{\mu}(n) = C u_{2n}$ for some absolute constant C, where u_{2n} is the probability that this walk returns to the origin at time $2n$. Again from ref. 11, Ch. 3, $u_{2n} \sim \text{const} \times n^{-1/2}$, which implies that BA, starting from μ, converges to its invariant state (which may be shown to be δ_{ϕ}) at the rate $time^{-1/2}$.

Example 2. CA 184, starting from Bernoulli 1/2 distribution, converges to the distribution $1/2(\delta_o + \delta_e)$ at the rate $time^{-1/2}$. This fact is obtained via mapping of CA 184 to BA and adapting to this process the argument from Example 1. Note that a certain adaptation is needed because the integrated profile in this case is not a simple random walk. All the details in respect may be found in work[13]. This work studied a so-called one-dimensional three-color cyclic cellular automaton. To show that this automaton converges to its invariant state at the rate $time^{-1/2}$, when starts from a particular distribution, the author maps it into BA. The initial distribution of the resulting BA is exactly the same as if it would have been obtained via $T_{184,\mathrm{BA}}$ from CA 184 that starts from Bernoulli 1/2 distribution.

5.3. An Application: The Long Time Limit of the Current of Particles in CA184

In this section, we define *current of particles* in CA 184, and show (Theorem 7) how it is calculated from parameters of initial measure of CA 184 in the case when this measure is translation invariant. In general, the particle current is of an interest, when CA 184 is used to model vehicular traffic. In particular, the ideas from Theorem 7 and its proof have been employed in ref. 2.

Let λ be a translation invariant measure on $\{0,1\}^{\mathbb{Z}}$. The *current (or flux) of particles* at time n in CA 184 $\{\eta_k\}_{k\in\mathbb{N}}$ starting from λ, is defined as

$$J_n(\lambda) := \lambda\{\eta_n(1)(1-\eta_{n-1}(1))\} = \lambda\{\eta_{n-1}(0)(1-\eta_{n-1}(1))\} \tag{65}$$

It expresses the expected number of particles that pass at time n through an observer put at the point 1/2; note that with this interpretation in mind, the justification of the last equality in (65) is straightforward. We shall be interested solely in the case when λ is translation invariant, so that the current does not depend on the observation point. When λ is also invariant for CA184 we shall write $J(\lambda)$ instead of $J_n(\lambda)$, since in this case the current does not depend on time. By

$$\rho(\lambda) := \lambda\{\eta(0)\} \tag{66}$$

we shall denote the *particle density* of a translation invariant measure λ. Besides the terms and notations just defined, Theorem 7 below will use the notations from Theorem 2.

Theorem 7. (a) (Particle current in CA184 distributed by its invariant measure that is transl. invariant.)

Let λ be a translation invariant measure on $\{0,1\}^{\mathbb{Z}}$ and let it be invariant for CA 184. Then

$$J(\lambda) = \begin{cases} \rho(\lambda), & \text{if } \lambda \in \mathbf{P}^{184}_{\tau,\text{hole blocks}} \\ 1-\rho(\lambda), & \text{if } \lambda \in \mathbf{P}^{184}_{\tau,\text{particle blocks}} \\ 1/2, & \text{if } \lambda = (\delta_{\text{even}} + \delta_{\text{odd}})/2 \end{cases} \tag{67}$$

If λ is none of the cases from (67) then

$$J(\lambda) = \alpha_1(1-\rho(\lambda_1)) + \alpha_2\rho(\lambda_2) + \frac{1}{2}\alpha_4 \tag{68}$$

where $\alpha_1, \alpha_2, \alpha_4$ and λ_1 and λ_2 are from expansion (36) from Theorem 2.

(b) (Long time limit of particle current in CA 184 starting from transl. invariant measure.) Let λ be any translation invariant measure on $\{0,1\}^{\mathbb{Z}}$. Consider its expansion as a convex combination of ergodic translation invariant measures on $\{0;1\}^{\mathbb{Z}}$. Define then $\lambda^{<1/2}$ (respectively, $\lambda^{>1/2}$ and $\lambda^{1/2}$) as the mixture, with the respective weights proportional to the original weights in the expansion of λ, of those of the measures from the expansion whose particle density is $<1/2$ (respectively, $>1/2$ and $=1/2$); let then α, β and γ be such that $\lambda = \alpha\lambda^{<1/2} + \beta\lambda^{>1/2} + \gamma\lambda^{1/2}$. Then the long time limit of the particle current in CA 184, starting from λ exists and is given by

$$J_\infty(\lambda) := \lim_{n\to\infty} J_n(\lambda) = \alpha\rho(\lambda^{<1/2}) + \beta(1-\rho(\lambda^{>1/2})) + \gamma\frac{1}{2} \tag{69}$$

Remark 8. As an example, let $\lambda = \nu/2 + \mu/2$ where ν and μ are two Bernoulli measures with respective particle densities 1/6 and 4/6. Note that a Bernoulli measure on $\{0;1\}^{\mathbb{Z}}$ is not invariant for CA 184. But since it is ergodic with respect to translation of $\mathbb{Z}$, then, from Theorem 7(b), we can conclude that the limit particle current in CA 184, starting from λ, is $1/2[1/6 + (1-4/6)] = 1/4$. Note that the latter is different from the particle density in CA 184 that is equal to $1/2(1/6+4/6) = 5/12$ at any (!) time (because particle neither appear nor disappear in CA 184).

The above example shows that in general, the particle density alone does not determine the particle current. There are however, two cases when it does. First, when λ is either $(\delta_{\text{even}} + \delta_{\text{odd}})/2$ or belongs to $\mathbf{P}^{184}_{\tau,\text{hole blocks}}$ or to $\mathbf{P}^{184}_{\tau,\text{particle blocks}}$; this case is covered by assertion (a) of Theorem 7. The second case is when λ is ergodic; this case gave raise to assertion (b). The role of the ergodicity of a measure in determining the particle current will be clarified in Remark 9.

Proof of (a). Consider first the case when either $\lambda \in \mathbf{P}^{184}_{\tau,\text{hole blocks}}$ or $\lambda = (\delta_{\text{even}} + \delta_{\text{odd}})/2$. Due to the definitions of $\mathbf{P}^{184}_{\tau,\text{hole blocks}}$, of δ_{even} and of δ_{odd} given in Theorem 2, such λ is supported by configurations in which particles cannot occupy adjacent sites, and thus, $\lambda\{\eta_n(i) = 0 \,|\, \eta_n(i-1) = 1\} = 1 \forall i \; \forall n$. This fact together with (65) and (66) imply that $J(\lambda) = \rho(\lambda)$. Consider now the case when $\lambda \in \mathbf{P}^{184}_{\tau,\text{hole blocks}}$. The definition (65) allows us to interpret $J(\lambda)$ as the expected number of holes that pass leftwards. Note that the rules of interactions of holes in CA 184 is the same as those of its particles (with the only difference that the drift directions are opposite). This fact allows us to adapt to the present case the argument of the above case, and to conclude that the current of holes in the leftward direction is equal to the density of holes. Since the latter is $1 - \rho(\lambda)$ then (67) is established.

The second statement in (a) follows from the first one, from Theorem 2 and from *additivity* of the particle current by which we mean the following property: if $\lambda = \alpha\mu + (1-\alpha)\nu$ for some $\alpha \in [0; 1]$ and some measures μ and ν, then $J_n(\lambda) = \alpha J_n(\mu) + (1-\alpha) J_n(\nu)$.

Let us prove (b). Pick arbitrarily $s \in (0; 1/2)$ and consider an ergodic translation invariant measure λ_s with the particle density equal to s. Since it is ergodic then

$$n^{-1} \lim_{n\to\infty} \sum_{i=0}^{n} \eta(i) = \rho(\lambda_s) = s \quad \lambda_s\text{-a.s.} \tag{70}$$

Consider CA 184 that starts from λ_s and denote by λ_s^∞ the limit distribution of this CA 184. Recall that Theorem 6(a) states that if BA starts from some translation invariant measure then it necessarily converges to a translation invariant measure that is invariant for its dynamics. Using the relation between BA and CA 184 given in Lemma 1 it is easy to deduce from this result that the same is true for CA 184 (this deduction is achieved exactly in the same manner as Theorem 2 has been deduced from Theorem 1). Thus, λ_s^∞ indeed exists, is translation invariant and invariant for CA 184.

Note now that CA 184 neither kills nor creates particles. Combining this fact with (70) we get that the particle density of each configuration in the support of λ_s^∞ is s.

Remark 9. It is here that we employ essentially the ergodicity of λ_s. It allows us to ensure that all the particle configurations have the same particle density. Otherwise, the particle density might have no relation to the current (as in the example from Remark 8), that would invalidate the argument below.

Since λ_s^∞ is translation invariant and invariant for CA 184, then, due to Theorem 2, it belongs to $\mathbf{P}^{184}_{\tau,\text{hole blocks}}$. By Theorem 7(a), then $J(\lambda_s^\infty) = \rho(\lambda_s^\infty)$, which is equal $\rho(\lambda_s)$, since, as we have noted, CA 184 neither kills nor creates particles. Thus, $\lim_{n\to\infty} J_n(\lambda_s) = \rho(\lambda_s)$.

Repeating the above argument, with obvious modifications, we get that $\lim_{n\to\infty} J_n(\lambda_s) = 1-\rho(\lambda_s)$ if $s > 1/2$ and is $1/2$ if $s = 1/2$.

Pick now an arbitrary translation invariant measure λ. Consider its expansion as a convex combination of ergodic measures: $\lambda = \sum_{e\in\mathcal{E}} p(e)\lambda(e)$; here $\mathcal{E}$ is an index set, each $\lambda(e)$ is an ergodic measure and $p(e)$ is its weight. From the conclusions just derived we have that

$$\lim_{n\to\infty} J_n(\lambda) = \sum_{e\in\mathcal{E}} \lim_{n\to\infty} p(e) J_n(\lambda(e)) = \sum_{e\in\mathcal{E}\,:\,\rho(\lambda(e))<1/2} p(e)\rho(\lambda(e)) + \sum_{e\in\mathcal{E}\,:\,\rho(\lambda(e))>1/2} p(e)(1-\rho(\lambda(e))) + \sum_{e\in\mathcal{E}\,:\,\rho(\lambda(e)=1/2} p(e)/2$$

from which the assertion (b) follows. ∎

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